Math 104, Summer 2019

Lecture 12, Thursday 7/11/2019

CLASS ANNOUNCEMENTS: Fluency in definitions (and rigor of theorems) is critical. You can surely do well via memorizing, but focusing on the key concepts should let you re-derive the theorems and definitions required.

Homework due Tuesday. Midterm solutions will be posted tonight, as well as homework. Grades will be out tomorrow (Friday) evening sometime, with summary statistics.

Grades will be raw scores out of possible points total, and then grade cut-offs will be generated at the end of the course.

Topics today:
- End of section 13;
- item

1 Review: The 'Closedness' Property

In a metric space S with metric d, open sets E are those for which

$$\forall x \in E, \exists_{r>0}\ B_r(x) \subseteq E,$$

where we define the **open** ball:

$$\{y \in S : d(x,y) < r\},$$

and x is an interior point of E (x is interior to E).
Analogously, $E \subseteq S$ is **closed** if and only if:

$$S \backslash E \text{ is open.}$$

And recall that we defined E^- as the 'closure' of E, the intersection of all closed sets containing E.

Remark: Intuitively, a closed set is one that includes all points in its boundary, whereas an open set is one that includes no points in its boundary. So a union of some closed set and some open set will most likely be neither closed nor open.

Also, recall from DeMorgan's Law:

$$S\backslash (E \cup F) = (S\backslash E) \cap (S\backslash F)$$
$$S\backslash (E \cap F) = (S\backslash E) \cup (S\backslash F)$$

Suppose E, F are both closed in the first line. Then the RHS is a finite intersection between open sets, so the whole RHS is open. Then we conclude $E \cup F$ is closed. That is,

$$S\backslash \underbrace{\left(\overbrace{E}^{\text{closed}} \cup \overbrace{F}^{\textit{closed}} \right)}_{\text{closed}} = \overbrace{\underbrace{(S\backslash E)}_{\text{open}} \cap \underbrace{(S\backslash F)}_{\text{open}}}^{\text{open}}$$

Theorem 1.1. Let $F_1, F_2, \ldots$ be nonempty, closed, bounded subsets of $\mathbb{R}^k$, with :

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$$

Then,

$$\cap_{n=1}^{\infty} F_n$$

is also nonempty, closed, and bounded.

Intuitively, the set in the intersection $\cap_{n=1}^{\infty}$ is the set of points in all of these F_i. There are three things we want to show about this intersection. To see closedness, it's trivial from DeMorgan's. Intersecting a closed set will always be closed (even if infinite intersections). Hence we take closedness to be known.
For boundedness, we can formalize via induction that our intersection is bounded.
What is nontrivial is that there must be some point in this intersection (the intersection is **nonempty**). To see this very briefly, take some sequence $(\vec{x}^{(n)})$ with each $\vec{x}^{(n)} \in F_n$. The fact that each of these F_i is each nonempty. Recall our generalization of Bolzano-Weierstrass to $\mathbb{R}^k$. By Bolzano-Weierstrass, there exists some $\left(\vec{x}^{(n_m)}\right)$ which is convergent subsequent limit will be in $\cap F_n$.

Example: Consider the following intersections:

$$\cap_n \left[0, \frac{1}{n}\right] = \{0\}$$
$$\cap_n \left[\frac{-1}{n}, \frac{1}{n}\right] = \{0\}$$
$$\cap_n \left(0, \frac{1}{n}\right) = \{\},$$

where the idea behind the failure to have an empty is not necessarily from openness but rather failure to be closed. Consider:

$$\cap_n \left(\frac{-1}{n}, \frac{1}{n}\right) = \{0\}.$$

Also, the Cantor Set:

$$\begin{aligned} F_1 :=& [0,1], \\ F_2 :=& \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], \\ F_3 :=& \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\ & \vdots \end{aligned}$$

Essentially we delete the middle 1/3 to get the next sequences.
The intersection of all of these: $\cap_n F_n$ is the Cantor set. This is a fractal; it is uncountable but has measure zero. We get a sequence that is descending,

and no point of the cantor set is isolated (no matter how small of a bubble we make in the set, there will always be points inside and outside of the bubble). The complement of the Cantor set (the boundary) is the cantor set (?).
We may see this later via the Cantor function, where the function is increasing, but the derivative wherever it exists is zero.

2 Compactness

Definition: Open cover -

Let (S, d) be a metric space. A family $\mathcal{U}$ of open sets is said to be an **open cover** for a set E if each point of E belongs to at least one set in $\mathcal{U}$. That is,

$$E \subseteq \bigcup \{U : U \in \mathcal{U}\}.$$

We say that a **subcover** U is any subfamily of U that also covers E. A cover or subcover is **finite** if it contains only finitely many sets; the sets themselves may be infinite.

Definition: Compact -

A set E is **compact** if *every* open cover of E *always* has a finite subcover of E.

In $\mathbb{R}^k$, for compact sets (definition of compact), we never need an infinite collection to cover a set. That is, there exists some finite collection that does this.

Example:

$$\{(n, n+2), \forall n \in \mathbb{Z}\} \text{ cover } \mathbb{R}.$$
$$\left\{\left(\frac{1}{n+2}, \frac{1}{n}\right), \forall n \in \mathbb{N}\right\} \text{ cover } (0,1).$$
$$\left\{\left(\frac{1}{n+2}, \frac{1}{n}\right) \forall n \in \mathbb{N}\right\} \bigcup \left\{\left(\frac{-1}{10}, \frac{1}{10}\right), \left(\frac{9}{10}, \frac{11}{10}\right)\right\} \text{ covers } [0,1].$$

The open cover does not need to 'exactly cover' a set; it only needs to inlude all the required points.
The first is not compact (we would expect $\mathbb{R}$ to be not compact). The second is still not compact, which has to do with the fact that it's open. In the third, our set is infinite, but it does have a finite subcover. Consider that the interval $\left(\frac{-1}{10}, \frac{1}{10}\right)$ includes all the other intervals on the left. Then we can cover the other points with the other bounds simply.
To show compactness, we want to show we can take any arbitrary open cover.
Consider $\mathcal{U}$ is always a subcover of itself. To prove something is **not** compact, it suffices to show an example of an infinite open set (?).
A subcover must still cover all of E, but it just happens to consist of subsets of the original cover, but it may not consist of all of U.

Theorem 2.1. Heine-Borel Theorem: A subset E of $\mathbb{R}^k$ is **compact** if and only if it is **closed** and **bounded**.

3 Series

We want to assign a value to an expression like this:

$$\sum_{k=m}^{\infty} a_k = a_m + a_{m+1} + a_{m+2} + \cdots$$

This expression does not inherently have meaning; however, we do want this to make sense.

Definition: Partial Sum -

We define s_n as a **partial sum** of the infinite sum above:

$$s_n := a_m + a_{m+1} + \cdots + a_n = \sum_{k=m}^{n} a_k.$$

Then we consider whether the sequence $(s_n)_{n=m}^{\infty}$ converges. Make sure not to confuse the two (original sequence, and the sequence of partial sums).
We also say that if $s_n \to \pm\infty$, then $\sum_{n=m}^{\infty} a_n = \pm\infty$.

Definition: 'Converges Absolutely' -

And if $\sum_{n=m}^{\infty} |a_n|$ converges, then we say

$$\sum_{n=m}^{\infty} a_n$$

converges absolutely. It follows as a consequence that $s_n := \sum_{n=m}^{\infty} a_n$ converges normally as well.

Known facts:

$$\sum_{n=0}^{\infty} = \frac{1}{2^n} = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-2}$$

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges iff } p > 1.$$

We say a series $\sum a_n$ sums satisfies the Cauchy criterion (is Cauchy) if its sequence (s_n) of partial sums is a Cauchy sequence. That is, for any $\epsilon > 0$, there exists a number N with $(m, n > N) \implies |s_n - s_m| < \epsilon$.
And equivalently,
For any $\epsilon > 0$, there exists a number N such that $n \geq m > N$ implies $|s_n - s_{m-1}| < \epsilon$.
But because $s_n - s_{m-1} = \sum_{k=m}^{n} a_k$, we can write this as:

Definition: Cauchy sequence (criterion) -

We say a series $\sum a_n$ sums satisfies the Cauchy criterion (is Cauchy) if its sequence (s_n) of partial sums is a Cauchy sequence. That is, for any $\epsilon > 0$, there exists a number N such that $n \geq m > N$ implies $|\sum_{k=m}^{n} a_k| < \epsilon$.

Theorem 3.1. If $\sum a_n$ converges, then we must have $a_n \to 0$ (the sequence of terms being summed approaches zero).

3.1 Comparison Tests

In most cases, we can't make a closed-form equation like we do have for $\sum a^n$. To prove convergence of these, we use comparison tests.

Theorem 3.2. Let $\sum a_n$ be a series where $a_n \geq 0$, for all n (all nonnegative terms). Then we have two things:

- If the series $\sum a_n$ converges and $|b_n| \leq a_n, \forall n$, then $\sum b_n$ **converges**.
- If $\sum a_n = +\infty$ and $b_n \geq a_n, \forall n$, then we say: $\sum b_n = +\infty$.

Theorem 3.3. Corollary: Absolutely convergent series are convergent.

Proof. Suppose $\sum b_n$ is absolutely convergent. This means $\sum a_n$ converges where $a_n = |b_n|$, for all n. Then by (i) above, by the comparison test, then $|b_n| \leq a_n$ trivially, so $\sum b_n$ converges. □

Theorem 3.4. Ratio Test (for Series)
A series $\sum a_n$ of nonzero terms has the following properties:

- if the ratio of terms $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ **converges absolutely**.
- if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ **diverges**.
- Otherwise, the test gives no information when $\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$.

Theorem 3.5. Root Test (for Series)
Let $\sum a_n$ be a series and let $\alpha := \limsup |a_n|^{1/n}$. Then the series $\sum a_n$:

- **converges absolutely** if $\alpha < 1$.
- **diverges** if $\alpha > 1$.
- Otherwise the test gives **no information** and $\alpha < 1$.

Remark: The root test is more powerful; every time that the ratio test would give a conclusive result, the root test would as well. However, it may be sometimes that the root tests are algebraically more intensive.

Lecture ends here. Next time we'll look into some examples of these tests.

4 Integral Test

Theorem 4.1. The sum $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.

Theorem 4.2. Alternating Series Theorem :
If $a_1 \geq a_2 \geq \cdots a_n \geq \cdots \geq 0$, and $\lim a_n = 0$, then the alternating series $\sum (-1)^{n+1} a_n$ converges. Moreover, the partial sums

$$s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$$

satisfy

$$\forall n, \quad |s - s_n| \leq a_n.$$