Math 104, Summer 2019
Lecture 1, Monday 6/24/2019

CLASS ANNOUNCEMENTS:
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OH: W/F 1-3pm 1047 Evans

Grading:
- Homework: 20% { due M,Th }
- MT1, MT2: 20% { Wed 7/10: 1.1-2.12, Tu 7/30: 2.13-3.22 }
- Final: 40% { Thu 8/15: comprehensive }

Homework Due Dates
- HW1: due Th 6/27
- HW2: due M 7/1
- HW3: due M 7/8
- HW4: due M 7/15
- HW5: due Th 7/18
- HW6: due M 7/22
- HW7: due Th 7/25
- HW8: due Th 8/1
- HW9: due M 8/5
- HW10: due Th 8/8
- HW11: due M 8/12
- HW12: due W 8/14

Remark: The general idea of the class: not computationally-intensive; more about what's going on under the hood when you perform calculus functions on x.

Goal today: Cover first 2 sections of Chapter 1 Ross.

1 About the Reals $\mathbb{R}$

What are real numbers? There are phenomena within the real line itself that defy human reason and within real-valued functions on the real line. The end of the book has a bit on the construction of real numbers and functions. What kinds of functions $\mathbb{R} \to \mathbb{R}$ exist?

Fundamentals of Calculus.
This course aims to consider questions like: what is the relationship between a function, its derivative, and its antiderivative?
We'll cover properties of: Continuity, Differentiability, Integrability
It can be helpful to bear in mind that Real Analysis has a goal or application of approximation.

1.1 Natural Numbers $\mathbb{N}$

Definition: σ successor -

The **successor** of n is $\sigma(n) = n + 1$.

Definition: Peano Axioms -

Peano Axioms:

1. $1 \in \mathbb{N}$
2. $(n \in \mathbb{N}) \implies ([n+1] \in \mathbb{N})$
3. $1 \notin \text{Im}(\sigma)$
4. $(\sigma(n) = \sigma(m)) \implies (n = m)$
5. (induction axiom) A subset of $\mathbb{N}$ which contains 1, and which contains $n+1$ whenever it contains n must equal $\mathbb{N}$ (must be the entire $\mathbb{N}$ itself)

To view how axiom 5 is important (and required), suppose it is false. Then $\mathbb{N}$ contains a set S such that: (1) $1 \in S$, and (2) If $n \in S$, then $(n+1) \in S$. But $S \neq \mathbb{N}$.

Then consider the smallest member of the set, $n_0 := \min\{n \in \mathbb{N} : n \notin S\}$. Because of (1), we must have $n_0 \neq 1$.

Then we have $(n_0 - 1) \in S$ because n_0 is the smallest element in $\mathbb{N}$ but NOT in S.

However, $\sigma(n_0 - 1) \in S$, but $\sigma(n_0 - 1) = n_0 - 1 + 1 = n_0 \notin S$ as established above, a contradiction.

Remark: Although this argument appears sound, we made two implicit UNPROVEN assumptions:

- every nonempty subset of $\mathbb{N}$ contains a least element
- if $n_0 \neq 1$, then n_0 is the successor to some number in $\mathbb{N}$

Example: Prove that $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$, for any $n \in \mathbb{N}$.

Proof. "Base case:" $n = 1$:

$$\frac{1}{2}(1)(1+1) = 1$$

"Inductive step": Suppose $1 + 2 + \cdots + k = \frac{1}{2}k(k+1)$.
If we show this implies $1 + 2 + \cdots + (k+1) = \frac{1}{2}[(k+1)][(k+1)+1]$, then we are done.

... (easy to see the result by adding $k+1$ to each side of our inductive hypothesis. □

Example: Prove that $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{N}$.

Proof. Consider the subset $U \subset \mathbb{N}$ for which this is true. We verify that for $n = 1$, $11^1 - 4^1 = 7$ is divisible by 7, so obviously $n \in U$. We also verify $2 \in U$ (because we use strong induction).

Assume that $k \in U$, so that $7a = 11^k - 4^k$ for some $a \in \mathbb{N}$. If we show this implies that

$$7|(11^{k+1} - 4^{k+1}),$$

then we are done.

Consider that

$$\begin{aligned} 11^{k+1} - 4^{k+1} &= 11(11^k) - 4(4^k) \\ &= 11(4^k + 7a) - 4(11^k - 7a) \quad \text{(substituting inductive hypothesis)} \\ &= 11(4^k) - 4^{11^k} + (7)(11)(a) + (4)(7)(a) \\ &= [(11)(4)] \cdot \underbrace{[4^{k-1} - 11^{k-1}]}_{\text{by strong induction, } k-1 \in U} + 7[11a + 4a] \end{aligned}$$

All terms are divisible by 7, so we have $(k-1) \in U \implies (k+1) \in U$, and are done.

$\square$

Definition: Integers -
$\mathbb{Z} := \{0, -1, 1, -2, 2, \dots\}$.

Definition: Rationals -
$\mathbb{Q} := \{\frac{m}{n}, \quad m, n \in \mathbb{Z}, n \neq 0\}$

1.2 Rational Numbers $\mathbb{Q}$

Pros: 'The space $\mathbb{Q}$ is a highly satisfactory algebraic system in which the basic operations (addition, multiplication, subtraction, and division) can be fully studied'.

Cons: But 'no system is perfect... and $\mathbb{Q}$ is inadequate in some ways'. For example, the diagonal of a square is irrational.

Definition: algebraic number -

A number x is called an **algebraic number** if it satisfies a polynomial equation

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where the coefficients $c_0, c_1, \dots, c_n \in \mathbb{Z}, \quad c_n \neq 0, \quad n \geq 1$.

Rational numbers $\mathbb{Q}$ are **always** algebraic numbers. To see this, $r = \frac{m}{n}, r \in \mathbb{Q}$ $(n \neq 0)$ gives $nr - m = 0$ (take $x := r$).

Numbers defined by $\sqrt{;}()^{1/3}$, etc, and ordinary algebraic operations on $\mathbb{Q}$ are invariably algebraic numbers.

Example: $\frac{4}{17}, \sqrt{3}, 17^{\frac{1}{3}}, \sqrt{\frac{4-2\sqrt{3}}{7}}$ are all algebraic numbers. To see this easily, consider

$$\begin{aligned} x &= \sqrt{\frac{4-2\sqrt{3}}{7}} \\ 7x^2 &= 4 - 2\sqrt{3} \\ 2\sqrt{3} &= 4 - 7x^2 \\ 12 - (4 - 7x^2)^2 &= 0 \\ 49x^4 - 56x^2 + (16 - 12) &= 0 \end{aligned}$$

Take-Home Problem: Show $\sqrt{2+(5)^{1/3}}$ is not rational.

Solution. Let $a := \sqrt{2+(5)^{1/3}}$. Then we have:

$$a^2 = [2+(5)^{1/3}] \quad \text{(squaring both sides)}$$
$$(a^2-1)^3 = 5$$

$\implies$ a is a root of $x^6 - 6x^4 + 12x^2 - 13 = 0$

The only possible rational roots (a) must have numerator the divides into ± 13 and denominator that divides into ± 1, by RZT. Plugging in these values shows that none of these work: $\{\pm 1, \pm 13\}$. ☐

Remark: To try if $x = 13$ is a root of $x^6 - 6x^4 + 12x^2 - 13 = 0$, we can see that $x = 13$ would give:

$$13\underbrace{(13^5 - 6\cdot 13^3 + 12\cdot 13 - 1)}\text{ is not } 0,$$

which we can see the right expression is not 0. Hence the root a cannot be rational.

Theorem 1.1. (Rational Zeros Theorem):

Suppose $c_0, c_1, \ldots, c_n \in \mathbb{Z}$ and $r \in \mathbb{Q}$, such that

$$c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r + c_0 = 0 \tag{1}$$

where $n \geq 1$, $\quad c_n \neq 0$, $\quad c_0 \neq 0$.

Let $r := \frac{c}{d}$ where $c, d \in \mathbb{Z}$, $d \neq 0$, and $\gcd(c,d) = 1$.

Then $c | c_0$ and $d | c_n$.

In other words, the only rational **candidates** for solutions of eqn (1) have the form $\frac{c}{d}$ where $c | c_0$ and $d | c_n$.

Proof. The hypotheses give us, for some $r = \frac{c}{d}$:

$$c_n\left(\frac{c}{d}\right)^n + c_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + c_1\left(\frac{c}{d}\right) + c_0 = 0$$

Multiply both sides by d^n to get:

$$c_n c^n + c_{n-1}c^{n-1}d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0$$

To see $c | c_0$, first solve for $c_0 d^n$:

$$c_0 d^n = -c\left[c_n c^{n-1} + c_{n-1}c^{n-2}d + \cdots + c_1 d^{n-1}\right]$$

We exhibited that c divides $c_0 d^n$, but $\gcd(c,d) = 1$, so c must divide into c_0.

To see $d | c_n$, we solve for $c_n c^n$:

$$c_n c^n = -d\left[c_{n-1}c^{n-1} + \cdots + c_1 c d^{n-2} + c_0 d^{n-1}\right]$$

We exhibited that d divides $c_n c^n$, but again, $\gcd(c,d) = 1$, so d must divide into c_n. ☐

Corollary 1.1.1. Consider "monic" polynomials of the form

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x_0 + c_0 = 0,$$

where the coefficients $c_0, \ldots, c_{n-1} \in \mathbb{Z}, c_0 \neq 0$. Any rational solution of this equation must be an integer that divides c_0.

Proof. This is simply a special case of the Rational Zeros Theorem above. In this scenario, $d|c_n = 1$, so $r = \frac{c}{d} = c \in \mathbb{Z}$. □

2 Axioms of $\mathbb{R}$

For all $a, b, c \in \mathbb{R}$,
A1. $a + (b + c) = (a + b) + c$; (assoc. of add)
A2. $a + b = b + a$ (commut. of add.)
A3. $a + 0 = a$ (exist. of add. ident. 0)
A4. For each a, $\exists -a$ with $a + (-a) = 0$. (add. inverse)
M1. $a(bc) = (ab)c$ (assoc. of mult.)
M2. $ab = ba$ (commut. of mult.)
M3. $a \cdot 1 = a$ (exist. of mult. identity 1)
M4. For each $a \neq 0, \exists a^{-1}$ with $aa^{-1} = 1$ (exist. of mult. inverse)
DL. $a(b + c) = ab + ac$ (distributive law)

The set $\mathbb{Q}$ has an order structure $\leq$ satisfying:

O1. Given a, b, either $a \leq b$ or $b \leq a$.
O2. $a \leq b, b \leq a \implies a = b$.
O3. $a \leq b, b \leq c \implies a \leq c$ (transitive law)
O4. $a \leq b \implies a + c \leq b + c$
O5. $a \leq b, 0 \leq c \implies ac \leq bc$

The following results come directly from the above axioms.

Theorem 3.1, Ross p.15 (Consequences of Field Properties)
(i) $a + c = b + c \implies a = b$;
(ii) $a \cdot 0 = 0, \quad \forall a$;
(iii) $(-a)b = -ab \quad \forall a, b$;
(iv) $(-a)(-b) = ab \quad \forall a, b$;
(v) $ac = bc, c \neq 0 \implies a = b$;
(vi) $ab = 0 \implies [(a = 0) \cup (b = 0)], \quad \forall a, b, c \in \mathbb{R}$.

Theorem 3.2, Ross p.16 (Consequences of an Ordered Field)

For all $a, b, c \in \mathbb{R}$, we have:
(i) $a \leq b \implies -b \leq -a$;
(ii) $a \leq b, c \leq 0 \implies bc \leq ac$;
(iii) $0 \leq a, 0 \leq b \implies 0 \leq ab$;
(iv) $0 \leq a^2, \quad \forall a$;
(v) $0 < 1$;
(vi) $0 < a \implies 0 < a^{-1}$;
(vii) $0 < a < b \implies 0 < b^{-1} < a^{-1}$;